JHEP HYPER VERSION

Received: *September 28, 2000,* Accepted: *November 06, 2000*
Revised: *November 03, 2000*

Topological defects in lattice gauge theories

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Abstract: We present a non-perturbative formalism for measuring defect free energies (monopole mass or vortex tension) in three-dimensional SU(2)+adjoint Higgs models. Starting from twisted, translation invariant boundary conditions, we perform a change of variables that allows us to express the defect free energies in terms of 't Hooft loops. We propose that the defect free energies can be used to distinguish between phases in this model, and also more generally in other gauge field theories where no local order parameters exist. In the case of monopoles, our construction can also be used in four-dimensional pure-gauge SU(2) theory, where it gives the monopole mass in the maximally abelian gauge without the need of actually fixing the gauge in the simulation. Moreover, the expression is manifestly independent of the choice of the abelian projection as long as it is compatible with the classical 't Hooft-Polyakov solution.

Keywords: Solitons Monopoles and Instantons, Spontaneous Symmetry Breaking, Lattice Gauge Field Theories, Cosmological Phase Transitions.

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Contents

1. Introduction

In many cases, phase transitions can be described in terms of topological defects. In some simple models, such as the Ising model [1] and super-Yang-Mills theories [2], an exact relation is known between this dual formulation and the original formulation of the theory. However, even without such an exact mapping, this dual description can be very useful, in particular since it gives a way of finding natural order parameters for the phase transition.

If the phase transition is associated with a spontaneous breakdown of a global symmetry, essentially any local operator that transforms non-trivially under the symmetry group can be used as an order parameter. However, if the broken symmetry is a gauge invariance, this approach does not work, because only gauge-invariant operators have non-zero expectation values. Although sometimes, for instance in the

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electroweak theory [3, 4], the transition predicted by perturbation theory is in fact only a smooth crossover, often a well-defined transition exists. In these cases, a dual picture in terms of topological defects can often be the most natural description.

There are two possible ways of constructing an order parameter based on topological defects. In theoretical studies, it is convenient to consider the creation operator μ of a defect [5]–[11]. In the low-temperature phase, its expectation value $\langle\mu\rangle$ vanishes, but if the defects condense in the high-temperature phase, it becomes non zero. Thus it behaves as an ordinary order parameter in a spontaneous symmetry breakdown, but with an inverted temperature, and often it can indeed be associated with a symmetry of the dual theory that is broken in the high-temperature phase. Therefore it is often called a disorder parameter [5]. However, this approach is only useful if the defects are line-like and can therefore be interpreted as world-lines of particles of the dual theory.

More generally, one can use the defect free energy [12]–[16] as an order parameter. In the special case of line-like defects, the tension, i.e. free energy per unit length, is simply the inverted correlation length of μ, and thus vanishes at the transition point. Using the defect free energy as the order parameter also avoids the problem that in numerical simulations $\langle\mu\rangle$ always vanishes because of the finite lattice size.

In the abelian Higgs model, Nielsen-Olesen vortices have been used successfully to study the phase transition. In refs. [6, 8], the disorder parameter and the corresponding U(1) symmetry were discussed, and in ref. [13] the tension of a vortex was measured and shown to be an order parameter. The purpose of this paper is to extend these studies to SU(2) theories with one or two adjoint Higgs fields by constructing explicitly the observables that measure free energies of vortices and monopoles.

Even though there is no spontaneous gauge symmetry breakdown in pure-gauge SU(2) theory, it is still believed that the non-perturbative dynamics of the gauge fields can give rise to similar effects. Both vortices [17] and monopoles [18] have been suggested as possible explanations for confinement. While the vortex tension can be expressed simply as the expectation value of a 't Hooft loop [17] and the procedure of measuring it is well understood [14, 15, 16], giving a non-perturbative definition for the monopole mass is more involved. It requires defining a composite adjoint field with an abelian projection [19], after which monopoles can be defined in the same way as in the Georgi-Glashow model [20, 21]. Disorder parameters based on monopoles have been studied in refs. [9, 10, 22], but they depend on the choice of the abelian projection, refer to solutions of classical field equations and don't have the same elegance as expressing the vortex tension in terms of a 't Hooft loop.

In this paper, we present a gauge invariant, non-perturbative formalism for studying vortices and monopoles in SU(2) theories with adjoint Higgs fields. In section 2, we discuss topological defects and defect free energies in general terms. Section 3 contains the derivation of a gauge-invariant expression for the residual U(1) gauge field in the broken phase. In section 4, we discuss the two-Higgs model and review the definition of the vortex tension in terms of a 't Hooft loop. This acts as an introduction to the definition of the monopole mass in the Georgi-Glashow model, which is carried out in section 5. In section 6, we show how the same construction can be used in pure-gauge theories with abelian projections.

2. Topological defects

In continuum, topologically stable non-trivial solutions of the field equations, i.e. topological defects, exist when the vacuum manifold of the system has non-trivial topology. They are characterized by a winding number, which is non zero if a defect is present. The mass of such a defect can be defined simply as the energy of the field configuration. When the symmetry is restored, the distinction between a defect and the vacuum disappears, and we can say that the mass of the defect vanishes.

However, in quantum systems and in classical systems at finite temperature, the state of the system is not given by a single field configuration, but rather by a density operator or an ensemble of configurations. In both cases, a useful description for the state of the system is given by the partition function, which has the form

$$Z = \int D\phi \exp(-S) \,. \tag{2.1}$$

Here $S = \int d^D x \mathcal{L}$ is the action and in the quantum case, we have performed the Wick rotation. In the thermal case, $S = \beta H$, but for simplicity we assume that the temperature is absorbed in the parameters of the lagrangian $\mathcal{L}$.

The partition function is a path integral over a large number of field configurations, almost none of which are solutions of the equations of motions. For each configuration, the winding number of the whole system is still a well-defined quantity. When it is zero, there can be localized objects that at a suitable length scale have the characteristics of a topological defect, but the total number of defects and anti-defects (those with a negative winding number) must be equal, and therefore they can be thought of as defect-antidefect pairs created by thermal fluctuations. However, if the total winding is non zero, the number of defects and anti-defects is not equal, which means that the configuration contains, in addition to the thermally generated defect-antidefect pairs, true topological defects. Note that in general it is not possible to determine which of the defects are the true ones and which are members of the thermally generated pairs.

From the point of view of local observables, it does not make any difference in the thermodynamical limit if we use instead of the canonical partition function (2.1) a microcanonical one Z_n, where the integration is restricted to configurations with a given total winding number n. However, the value of Z_n is different for each n. In fact, it is more useful to consider the free energy

$$F_n = -\ln Z_n \,, \tag{2.2}$$

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because in the classical limit where the integral is dominated by its saddle point, the free energy difference

$$\Delta F = F_1 - F_0 \tag{2.3}$$

is exactly the classical mass of the defect. We will call ΔF the free energy of a defect. Note that it is defined without recourse to the saddle-point approximation.

It is not obvious how to use eq. (2.3) in lattice field theories, because the topology of the field configuration space is very different in a discrete space. For instance, the abelian Higgs model has two different lattice formulations, the compact and the non-compact one, and only the non-compact version contains topological defects. Thus we will first have to show that in any particular case we are discussing, the winding number can really be defined. Even that is not enough, because in lattice simulations, it is only possible to measure expectation values

$$\langle \mathcal{O} \rangle = Z^{-1} \int D\phi \mathcal{O} \exp(-S) \,, \tag{2.4}$$

and not free energies directly. Therefore, we will also have to rewrite eq. (2.3) in terms of expectation values.

3. Residual U(1) invariance

Let us start by considering in general an SU(2) gauge theory with an adjoint scalar field Φ, which may be either fundamental or composite. We will now derive an expression for the abelian gauge field corresponding to the residual U(1) gauge invariance that remains if the field Φ breaks the SU(2) symmetry. A similar construction has previously been carried out in continuum [20] and on a lattice in refs. [23, 24].

In this section, we assume that the system is defined on a three-dimensional lattice, which may also be a single time slice of a four-dimensional lattice. The gauge field is represented by SU(2) matrices $U_i(\vec{x})$ defined on the links $(\vec{x}, \vec{x}+\hat{\imath})$ between the lattice sites $\vec{x} \in \{0, \ldots, N-1\}^3$, and the adjoint scalar field Φ is defined on lattice sites.

The theory is invariant under gauge transformations

$$\begin{aligned} \Phi(\vec{x}) &\longrightarrow \Lambda^\dagger(\vec{x})\Phi(\vec{x})\Lambda(\vec{x}) \,, \\ U_i(\vec{x}) &\longrightarrow \Lambda^\dagger(\vec{x})U_i(\vec{x})\Lambda(\vec{x}+\hat{\imath}) \,, \end{aligned} \tag{3.1}$$

where $\Lambda(\vec{x})$ is an SU(2)-valued function defined on the lattice sites. In particular, it is always possible to gauge transform Φ to the z-direction.

To simplify the discussion, we neglect the configurations in which Φ either vanishes or is proportional to σ_3 at any lattice site. Because they have a zero measure in the partition function, this does not change the results, and our final results work well even in these special cases. This allows us to define the unit vector

$$\hat{\Phi} = \Phi(\Phi^2)^{-1/2} \,. \tag{3.2}$$

We start the construction by making a gauge transformation that diagonalizes Φ, i.e. turns $\hat{\Phi}$ into σ^3 at every point. That is accomplished by

$$R(\vec{x}) \propto i(\sigma^3 + \hat{\Phi}(\vec{x}))\,. \tag{3.3}$$

Here and in the following we use the notation $\propto$ to show an equality that is true up to a real and positive factor. Our final expressions will be independent of these factors. In eq. (3.3), the normalization is chosen in such a way that R is an SU(2) matrix.

We can now define the transformed gauge field

$$\tilde{U}_i(\vec{x}) = R^\dagger(\vec{x}) U_i(\vec{x}) R(\vec{x}+\hat{\imath})\,. \tag{3.4}$$

Since the corresponding value of Φ is proportional to σ^3, R is nothing but a gauge transformation into the unitary gauge. However, this does not fix the gauge completely, because of the residual U(1) invariance. More precisely, a gauge transformation with a matrix Λ induces a transformation

$$\tilde{U}_i(\vec{x}) \longrightarrow \tilde{\Lambda}^\dagger(\vec{x}) \tilde{U}_i(\vec{x}) \tilde{\Lambda}(\vec{x}+\hat{\imath})\,, \tag{3.5}$$

where

$$\tilde{\Lambda} = R^\dagger \Lambda R_\Lambda\,, \tag{3.6}$$

and

$$R_\Lambda \propto i(\sigma^3 + \Lambda^\dagger \hat{\Phi} \Lambda)\,. \tag{3.7}$$

The new transformation $\tilde{\Lambda}$ is unitary,

$$\tilde{\Lambda}^\dagger \tilde{\Lambda} = R_\Lambda^\dagger \Lambda^\dagger R R^\dagger \Lambda R_\Lambda = 1\,, \tag{3.8}$$

and its determinant is one. It is also diagonal, since it leaves σ^3 invariant by construction, and therefore it must be of the form $\tilde{\Lambda} = \exp(i\lambda\sigma^3)$. This is the residual abelian gauge transformation.

Let us then show that the corresponding gauge field is given by the phase of the diagonal elements of $\tilde{U}_i$. We define the projection operators

$$P_\pm = \frac{1}{2}(\mathbb{I} \pm \sigma^3)\,, \qquad \text{i.e.} \qquad P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

and the projected fields

$$\tilde{V}_{i,\pm}(\vec{x}) = P_\pm \tilde{U}_i(\vec{x}) P_\pm\,. \tag{3.10}$$

The field $\tilde{V}_{i,+}$ ($\tilde{V}_{i,-}$) corresponds to the upper left (lower right) component of the total gauge field $\tilde{U}_i$. Furthermore, if we define

$$\Pi_\pm = \frac{1}{2}(\mathbb{I} \pm \hat{\Phi})\,, \tag{3.11}$$

we can write $\tilde{V}_{i,\pm}(\vec{x}) = R^\dagger(\vec{x}) V_{i,\pm}(\vec{x}) R(\vec{x}+\hat{\imath})$, where

$$V_{i,\pm}(\vec{x}) = \Pi_\pm(\vec{x}) U_i(\vec{x}) \Pi_\pm(\vec{x}+\hat{\imath})\,. \tag{3.12}$$

Under a gauge transformation, the fields $\tilde{V}_{i,\pm}$ transform as

$$\tilde{V}_{i,\pm}(\vec{x}) \longrightarrow \tilde{\Lambda}^\dagger(\vec{x}) \tilde{V}_{i,\pm}(\vec{x}) \tilde{\Lambda}(\vec{x}+\hat{\imath}) = \exp\left\{\pm i \left(\lambda(\vec{x}+\hat{\imath}) - \lambda(\vec{x})\right)\right\} \tilde{V}_{i,\pm}(\vec{x})\,, \tag{3.13}$$

which means that

$$\alpha_i \equiv \arg \mathrm{Tr}\, \tilde{V}_{i,+} = - \arg \mathrm{Tr}\, \tilde{V}_{i,-} \tag{3.14}$$

indeed behaves like the abelian gauge field. Thus we can also define the magnetic flux density as

$$\begin{aligned} \alpha_{ij} &\equiv \arg \mathrm{Tr}\, \tilde{V}_{i,+}(\vec{x}) \tilde{V}_{j,+}(\vec{x}+\hat{\imath}) \tilde{V}^\dagger_{i,+}(\vec{x}+\hat{\jmath}) \tilde{V}^\dagger_{j,+}(\vec{x}) \\ &= \arg \mathrm{Tr}\, \Pi_+(\vec{x}) U_i(\vec{x}) \Pi_+(\vec{x}+\hat{\imath}) U_j(\vec{x}+\hat{\imath}) \Pi_+(\vec{x}+\hat{\imath}+\hat{\jmath}) U_i^\dagger(\vec{x}+\hat{\jmath}) \Pi_+(\vec{x}+\hat{\jmath}) U_j^\dagger(\vec{x})\,. \end{aligned} \tag{3.15}$$

This final version is gauge invariant and contains no reference to the fields in the unitary gauge.

Using eq. (3.15), we can define the magnetic charge inside a lattice cube as

$$C_M(\vec{x}) = \frac{1}{4\pi} \sum_{ijk} \epsilon_{ijk} \left(\alpha_{jk}(\vec{x}+\hat{\imath}) - \alpha_{jk}(\vec{x})\right)\,. \tag{3.16}$$

This number is an integer and can be non zero.

4. 3D two-Higgs model

Let us first consider a model with two adjoint Higgs fields, Φ and χ. The lagrangian is

$$\begin{aligned} \mathcal{L}_{2\mathrm{H}} = \frac{4}{ag^2} \sum_{i<j} &\left(1 - \frac{1}{2} \mathrm{Re}\ \mathrm{Tr}\, U_i(x) U_j(\vec{x}+\hat{\imath}) U_i^\dagger(\vec{x}+\hat{\jmath}) U_j^\dagger(x)\right) + \\ &+ \sum_i 2a \Big[\mathrm{Tr}\, \Phi^2(\vec{x}) - \mathrm{Tr}\, \Phi(\vec{x}) U_i(\vec{x}) \Phi(\vec{x}+\hat{\imath}) U_i^\dagger(\vec{x}) + \\ &\qquad + \mathrm{Tr}\, \chi^2(\vec{x}) - \mathrm{Tr}\, \chi(\vec{x}) U_i(\vec{x}) \chi(\vec{x}+\hat{\imath}) U_i^\dagger(\vec{x})\Big] + \\ &+ m_\Phi^2 a^3\, \mathrm{Tr}\, \Phi^2(\vec{x}) + a^3 \lambda_\Phi (\mathrm{Tr}\, \Phi^2)^2 + \\ &+ m_\chi^2 a^3\, \mathrm{Tr}\, \chi^2(\vec{x}) + a^3 \lambda_\chi (\mathrm{Tr}\, \chi^2)^2 + a^3 \eta\, \mathrm{Tr}\, \Phi^2 \chi^2 + a^3 \eta_2 \left(\mathrm{Tr}\, \Phi\chi\right)^2\,. \end{aligned} \tag{4.1}$$

If Φ is non zero, the two-Higgs model looks very much like the abelian Higgs model. In figure 1 we show the plot of the susceptibility of $\mathrm{Tr}\, \chi^2$ at the point where χ becomes non zero in perturbation theory (see the appendix for numerical details). The maximum susceptibility does not increase with the volume, and therefore the

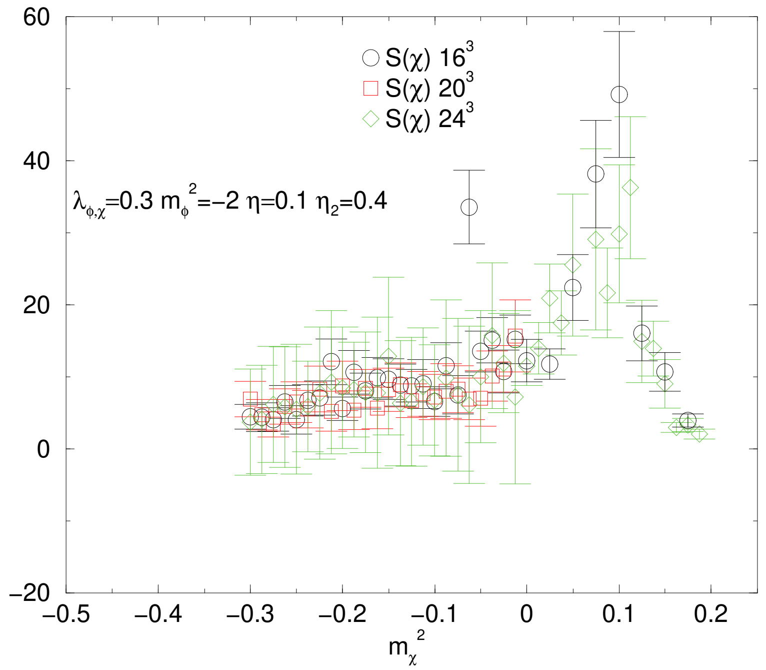

Figure 1: The susceptibility of $\mathrm{Tr}\,\chi^2$ near the perturbative transition point.

field $\mathrm{Tr}\,\chi^2$ does not behave non-analytically. The same applies to any local observable, and therefore we are forced to consider non-local order parameters, and in particular the defect free energy (2.3). Because in this model, the defects are line-like vortices, it is convenient to define the tension as the free-energy per unit length of a vortex

$$T = \lim_{L\to\infty} \frac{\Delta F}{L}\,. \tag{4.2}$$

The construction of this order parameter is well known and dates back to late seventies [17, 25], but we will rederive it here, because the derivation of the monopole mass in the one-Higgs model follows similar lines.

4.1 Winding number

We start by deriving the lattice expression for the winding number. We will follow the corresponding derivation in the case of the abelian Higgs model [26] (see also ref. [24]). Since we already know the analogue (3.14) of the abelian gauge field, we only have to find the corresponding Higgs field. If we define using eq. (3.3)

$$\tilde{\chi} = R^\dagger \chi R\,, \tag{4.3}$$

we find that it transforms as

$$\tilde{\chi} \longrightarrow \tilde{\Lambda}^\dagger \tilde{\chi} \tilde{\Lambda}\,. \tag{4.4}$$

If we now project to the lower left matrix element of $\tilde{\chi}$

$$h = P_- \tilde{\chi} P_+\,, \tag{4.5}$$

we find the transformation law

$$h = P_- R\chi R P_+ = R\Pi_-\chi\Pi_+ R \longrightarrow R_\Lambda \Lambda^\dagger \Pi_-\chi\Pi_+\Lambda R_\Lambda = \tilde{\Lambda}^\dagger h \tilde{\Lambda} = \exp(2i\lambda) h\,. \quad (4.6)$$

Thus h behaves as a charged scalar field. Note, however, that the charge of h is two, and therefore the transformation $\lambda = \pi$ leaves it unchanged. This transformation is the remaining $\mathbb{Z}_2$ symmetry that is left after h gets an expectation value.

Now, we need a gauge invariant expression for the difference of the Higgs phase angle at neighbouring lattice sites

$$\begin{aligned}\Delta_i &= \mathrm{Tr}\, h^\dagger(\vec{x})\tilde{V}_{i,-}(\vec{x})h(\vec{x}+\hat{\imath})\tilde{V}^\dagger_{i,+}(\vec{x}) \\ &= \mathrm{Tr}\,\hat{\chi}(\vec{x})\Pi_-(\vec{x})U_i(\vec{x})\Pi_-(\vec{x}+\hat{\imath})\hat{\chi}(\vec{x}+\hat{\imath})\Pi_+(\vec{x}+\hat{\imath})U^\dagger_i(\vec{x})\Pi_+(\vec{x})\,. \end{aligned} \quad (4.7)$$

Defining the Higgs phase angle γ by $h \propto \exp(i\gamma(\vec{x}))$ and using eq. (3.14), we find (cf. ref. [26, eq. (10)])

$$\delta_i \equiv \arg \Delta_i = [\gamma(\vec{x}+\hat{\imath}) - \gamma(\vec{x}) - 2\alpha_i(\vec{x})]_\pi\,, \quad (4.8)$$

where we use the notation $[X]_\pi \equiv X + 2\pi n_X$, with such an n_X that $[X]_\pi \in (-\pi,\pi]$.

To define the winding number we have to subtract the magnetic flux. This leads to the expression

$$Y_{ij}(x) = \delta_i(x) + \delta_j(\vec{x}+\hat{\imath}) - \delta_i(\vec{x}+\hat{\jmath}) - \delta_j(x) - 2\alpha_{ij}(\vec{x})\,, \quad (4.9)$$

which is obviously a multiple of 2π and thus the winding number $n_{ij} \equiv (2\pi)^{-1}Y_{ij}$ is always an integer. The value of Y_{ij} is only defined modulo 4π, and therefore one can only distinguish between even and odd values of n_{ij}. If it is odd, there is a vortex going through the plaquette.

The definition of the winding number can be extended to more complicated curves than just single plaquettes in a straightforward way by adding all the δ's and α_i's along the curve, and the result is additive

$$n_{C+C'} = n_C + n_{C'} \bmod 2\,. \quad (4.10)$$

This equation implies that vortices cannot end.

4.2 Vortex tension

Now that we have the expression for the winding number, we have to restrict the path integral to configurations with a given winding. The most natural way of doing it is by boundary conditions. In lattice simulations, the boundary conditions are almost always chosen to be periodic,

$$\Phi(\vec{x}+N\hat{\jmath}) = \Phi(\vec{x})\,, \qquad \chi(\vec{x}+N\hat{\jmath}) = \chi(\vec{x})\,, \qquad U_k(\vec{x}+N\hat{\jmath}) = U_k(\vec{x})\,. \quad (4.11)$$

This implies periodicity also for δ_i and α_i, and therefore the total winding is even, i.e. there are no vortices.

Besides their simplicity, one reason for using the periodic boundary conditions is that they minimize the finite-size effects, since they guarantee that the actual boundaries will be invisible to the physics. This is particularly important when a defect free energy is measured, because if there are physical boundaries on the lattice, their contribution to the free energy can dominate over it. However, any boundary conditions that preserve the translation invariance of the system avoid these problems, and therefore it is enough to have boundary conditions that are periodic up to any symmetries of the theory. We can, for instance, modify the boundary conditions (4.11) by

$$U_2(x+N,y,z)=\begin{cases}-U_2(x,y,z)\,, & \text{if } y=1\,,\\ U_2(x,y,z), & \text{if } y\neq 1\,.\end{cases} \tag{4.12}$$

Changing the sign of $U_2(x,1,z)$ does not change the lagrangian (4.1) and therefore there are no boundary effects. Moreover, δ_i are still periodic, since they always contain U_i as well as $U_i^\dagger$. However, $\alpha_y(x,1,z)$ changes by π, and if we calculate the winding of the whole xy plane according to eq. (4.9), $\alpha_y(x,1,z)$ appears once, and thus the total winding changes from even to odd. This shows that these non-periodic boundary conditions lead to configurations with one vortex in z-direction. The free energies in eq. (2.3) are thus given by

$$F_n=-\ln Z=-\ln\int\mathcal{D}U_i\mathcal{D}\Phi\mathcal{D}\chi\exp\left(-\sum_{\vec{x}}\mathcal{L}_{2\mathrm{H}}\right), \tag{4.13}$$

and the subscript $n\in\{0,1\}$ indicates which of the boundary conditions (4.11) and (4.12) are used.

Next, we have to write $T=\Delta F/L$ in terms of expectation values. Let us first notice that when we calculate the action $S=\sum\mathcal{L}_{2\mathrm{H}}$, the boundary condition (4.12) only affects the Wilson term. Instead of saying that we are using non-periodic boundary conditions, we could say that the boundary conditions are periodic, but the action is changed by

$$\Delta S=\frac{4}{ag^2}\sum_z\mathrm{Tr}\,U_1(N,1,z)U_2(1,1,z)U_1^\dagger(N,2,z)U_2^\dagger(N,1,z)\,. \tag{4.14}$$

This corresponds to a change in the integration variable in the path integral, which should be taken into account when measuring any observable, but since we are now only interested in the tension, we do not have to worry about that. The translation invariance of the boundary conditions implies that we could equally well choose any coordinates x_0, y_0 and write

$$\Delta S=\frac{4}{ag^2}\sum_z\mathrm{Tr}\,U_{12}(x_0,y_0,z)\,. \tag{4.15}$$

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This allows us to write

$$T = -\lim_{L\to\infty} \frac{\ln\langle\exp(-\Delta S)\rangle}{L}\,. \tag{4.16}$$

This form is not, however, particularly suitable for numerical simulations, because the observable $\exp(-\Delta S)$ and the integration weight $\exp(-S)$ have only very little overlap. One solution would be to measure derivatives of T. For instance,

$$\frac{\partial T}{\partial m_\chi^2} = a^2N^2\left(\langle \mathrm{Tr}\,\chi^2\rangle_1 - \langle \mathrm{Tr}\,\chi^2\rangle_0\right), \tag{4.17}$$

where the subscript indicates whether the expectation value is calculated in the one-vortex or the zero-vortex ensemble. this can then be integrated to yield T. The drawback of this method is that it cannot be used to measure the absolute value of T, only its differences.

Another possibility is to introduce a real number $\epsilon \in [0,1]$ and define non-physical ensembles with the action $S+\epsilon\Delta S$. We can define

$$F_\epsilon = -\ln\int \mathcal{D}U_i\mathcal{D}\Phi\mathcal{D}\chi\exp(-S-\epsilon\Delta S)\,, \tag{4.18}$$

and it is then easy to write the tension as

$$T = \frac{1}{L}\int_0^1 d\epsilon\frac{dF_\epsilon}{d\epsilon} = \frac{1}{L}\int_0^1 d\epsilon\langle\Delta S\rangle_\epsilon\,, \tag{4.19}$$

where $\langle\cdots\rangle_\epsilon$ means an expectation value calculated using the action $S+\epsilon\Delta S$. The tension can now be calculated in numerical simulations by measuring the integrand of eq. (4.19) at a large number of different values of ϵ. Note that although ensembles with non-integer ϵ are not physical, the final result T is. This method can be improved by using multi-histogram techniques [14, 27, 28]. An integration scheme in which $\epsilon = 1$ but plaquettes are added to eq. (4.15) one by one has also been suggested recently [16].

The shift (4.15) of the action is a 't Hooft loop [17, 25] that is made stable by closing it via the periodic boundary conditions. Its relation to the twisted boundary conditions (4.12) was first discussed in refs. [29, 30].

In pure-gauge SU(2), the vortex tension has been measured in refs. [14, 15, 16]. Instead of having a closed loop, the sum over z in eq. (4.15) can also be restricted to a shorter interval $\{z_i,\ldots,z_f\}$

$$\langle\bar{\mu}(z_f)\mu(z_i)\rangle = Z^{-1}\int \mathcal{D}U_i\mathcal{D}\Phi\mathcal{D}\chi\exp\left(-S-\frac{4}{ag^2}\sum_{z=z_i}^{z_f}\mathrm{Tr}\,U_{12}(x_0,y_0,z)\right). \tag{4.20}$$

This technique was used in ref. [28] to study the monopole-antimonopole interaction in the pure-gauge SU(2) theory. If the space is interpreted as $2+1$ dimensional and the z direction as time, this describes a creation of a vortex at time z_i and its annihilation at time z_f [31]. This leads to a dual picture of the theory in which the fundamental degrees of freedom are vortices and the phase transition has an interpretation of a spontaneous symmetry breakdown of a magnetic symmetry for which $\langle\mu\rangle$ defined by

$$\langle\mu\rangle^2 = \lim_{z\to\infty}\langle\bar{\mu}(z)\mu(0)\rangle \tag{4.21}$$

is an order parameter.

5. 3D Georgi-Glashow model

Let us now consider a system with only one adjoint Higgs field. We will use the lagrangian

$$\begin{aligned}\mathcal{L}_{1\mathrm{H}} = {} & \frac{4}{ag^2}\sum_{i<j}\left(1-\frac{1}{2}\mathrm{Re}\ \mathrm{Tr}\, U_{ij}(\vec{x})\right)+\\ & +\sum_i 2a\left[\mathrm{Tr}\,\Phi^2(\vec{x})-\mathrm{Tr}\,\Phi(\vec{x})U_i(\vec{x})\Phi(\vec{x}+i)U_i^\dagger(\vec{x})\right]+\\ & +m^2a^3\,\mathrm{Tr}\,\Phi^2(\vec{x})+a^3\lambda(\mathrm{Tr}\,\Phi^2)^2\,.\end{aligned} \tag{5.1}$$

The topological defects in this system are the 't Hooft-Polyakov monopoles [20, 21], which are point-like objects. Thus we can define the mass of a monopole using eq. (2.3) simply as

$$M = \Delta F\,. \tag{5.2}$$

Mean field theory predicts that monopoles are massive in the broken phase and massless in the symmetric phase. Therefore the mass would be a useful order parameter for the phase transition. However, the masslessness of monopoles in the three-dimensional theory would not imply that there are infinite correlation lengths in the symmetric phase, because a monopole is a point-like object and therefore its mass is not related to the correlation length of any operator of the dual theory. In four dimensions, creation and annihilation operators of monopoles have been discussed in ref. [11] using a different approach.

5.1 Boundary conditions

Again, our strategy is to find boundary conditions that preserve the translation invariance of the system but force the total magnetic charge of the lattice to be one. Periodic boundary conditions

$$\Phi(\vec{x}+N\hat{\jmath}) = \Phi(\vec{x})\,,\qquad U_k(\vec{x}+N\hat{\jmath}) = U_k(\vec{x})\,, \tag{5.3}$$

are obviously ruled out, because they don't allow non-zero total charge. Instead, we need more complicated boundary conditions, and we can use as a guidance the classical continuum monopole solution

$$\Phi(\vec{x}) \approx \frac{x_k \sigma_k}{r}\,, \qquad A_i(\vec{x}) \approx \frac{\epsilon_{ijk} x_j \sigma_k}{2r^2}\,. \tag{5.4}$$

If we move from one boundary to another, we reverse the sign of the coordinate x_j, and the fields transform as

$$x_j \longrightarrow -x_j : \qquad \Phi \longrightarrow -\sigma_j \Phi \sigma_j\,, \qquad A_i \longrightarrow \sigma_j A_i \sigma_j\,. \tag{5.5}$$

This suggests the boundary conditions

$$\Phi(\vec{x}+N\hat{\jmath}) = -\sigma_j \Phi(\vec{x}) \sigma_j\,, \qquad U_k(\vec{x}+N\hat{\jmath}) = \sigma_j U_k(\vec{x}) \sigma_j\,. \tag{5.6}$$

It is straightforward to see that this is a symmetry of the lagrangian (5.1), and therefore does not break translation invariance.

The boundary conditions (5.6) imply that $\Pi_+(\vec{x}+N\hat{\jmath}) = \sigma_j \Pi_-(\vec{x})\sigma_j$ and, consequently,

$$\alpha_{ij}(\vec{x}+N\hat{k}) = -\alpha_{ij}(\vec{x})\,, \tag{5.7}$$

i.e. the direction of the magnetic flux is reversed at the boundaries. Thus, when we cross the boundary of the lattice, we enter a charge-conjugated copy of the same lattice from the opposite boundary. This is however not enough to guarantee that the total charge of each of these copies is non zero.

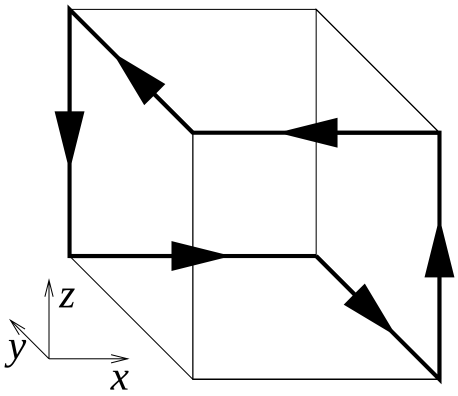

Figure 2: The curve used in eq. (5.8) to separate the boundary of the lattice into two halves.

To determine the total charge, we calculate the magnetic flux through half of the boundary. The curve defined in figure 2 separates the boundary into two halves, and we can calculate

$$\text{flux} = \arg \mathrm{Tr} \prod_{\text{curve}} \tilde{V}_{i,+}(\vec{x}) = \arg \prod_{ijk\ \text{cyclic}} \prod_{n=0}^{N-1} \mathrm{Tr}\, \tilde{V}_{i,+}(n\hat{\imath}+N\hat{\jmath}) \tilde{V}^{\dagger}_{i,+}(n\hat{\imath}+N\hat{k})\,, \tag{5.8}$$

where the final form follows from the fact that the $\tilde{V}_{i,+}$ commute. To calculate this, we need to know the boundary conditions induced by eq. (5.6) for R:

$$\begin{aligned} R(\vec{x}+N\hat{x}) &= -\sigma_1 R(\vec{x})\sigma_1\,, \\ R(\vec{x}+N\hat{y}) &= -\sigma_2 R(\vec{x})\sigma_2\,, \\ R(\vec{x}+N\hat{z}) &\propto \sigma_3 R(\vec{x})[\hat{\Phi}(\vec{x}),\sigma_3]\sigma_3\,. \end{aligned} \tag{5.9}$$

Then it is easy to calculate the corresponding boundary conditions for $\tilde{V}_{i,\pm}$:

$$\begin{aligned}
\tilde{V}_{i,+}(\vec{x}+N\hat{x}) &= \sigma_1 \tilde{V}_{i,-}(\vec{x})\sigma_1\,, \\
\tilde{V}_{i,+}(\vec{x}+N\hat{y}) &= \sigma_2 \tilde{V}_{i,-}(\vec{x})\sigma_2\,, \\
\tilde{V}_{i,+}(\hat{x}+N\hat{z}) &\propto \sigma_3[\sigma_3,\hat{\Phi}(\hat{x})]\tilde{V}_{i,-}(\hat{x})[\hat{\Phi}(\vec{x}+\hat{\imath}),\sigma_3]\sigma_3\,.
\end{aligned} \tag{5.10}$$

The case $i=3$ does not contribute, because

$$\begin{aligned}
\mathrm{Tr}\,\tilde{V}_{3,+}(n\hat{z}+N\hat{x})\tilde{V}^{\dagger}_{3,+}(n\hat{z}+N\hat{y}) &= \mathrm{Tr}\,\sigma_1\tilde{V}_{3,-}(n\hat{z})\sigma_1\sigma_2\tilde{V}^{\dagger}_{3,-}(n\hat{z})\sigma_2 \\
&= \mathrm{Tr}\,\tilde{V}_{3,-}(n\hat{z})\tilde{V}^{\dagger}_{3,-}(n\hat{z}) = 1\,,
\end{aligned} \tag{5.11}$$

where we have used the fact that $\sigma_1\sigma_2 P_\pm = i\sigma_3 P_\pm = \pm i P_\pm$. For the horizontal segments we find

$$\begin{aligned}
\mathrm{Tr}\,\tilde{V}_{1,+}(\vec{x}+N\hat{y})\tilde{V}^{\dagger}_{1,+}(\vec{x}+N\hat{z}) &\propto \mathrm{Tr}\,\tilde{V}_{1,-}(\vec{x})\sigma_1[\sigma_3,\hat{\Phi}(\vec{x}+\hat{x})]\tilde{V}^{\dagger}_{1,-}(\vec{x})[\hat{\Phi}(\vec{x}),\sigma_3]\sigma_1\,, \\
\mathrm{Tr}\,\tilde{V}_{2,+}(\vec{x}+N\hat{z})\tilde{V}^{\dagger}_{2,+}(\vec{x}+N\hat{x}) &\propto \mathrm{Tr}\,\sigma_2[\sigma_3,\hat{\Phi}(\vec{x})]\tilde{V}_{2,-}(\vec{x})[\hat{\Phi}(\vec{x}+\hat{y}),\sigma_3]\times \\
&\quad\times\sigma_2\tilde{V}^{\dagger}_{2,-}(\vec{x})\,,
\end{aligned} \tag{5.12}$$

where we have written $n\hat{\imath}=\vec{x}$ for notational simplicity. Because

$$\begin{aligned}
\sigma_1[\sigma_3,\hat{\Phi}] &= 2\begin{pmatrix} -\hat{\Phi}_1 - i\hat{\Phi}_2 & 0 \\ 0 & \hat{\Phi}_1 - i\hat{\Phi}_2 \end{pmatrix}, \\
\sigma_2[\sigma_3,\hat{\Phi}] &= 2i\begin{pmatrix} \hat{\Phi}_1 + i\hat{\Phi}_2 & 0 \\ 0 & \hat{\Phi}_1 - i\hat{\Phi}_2 \end{pmatrix}.
\end{aligned} \tag{5.13}$$

and because $\tilde{V}_{i,-}$ projects to the lower right component, we obtain

$$\begin{aligned}
\mathrm{Tr}\,\tilde{V}_{1,+}(n\hat{x}+N\hat{y})\tilde{V}^{\dagger}_{1,+}(n\hat{x}+N\hat{z}) &\propto \exp\{-i\,[\theta((n+1)\hat{x})-\theta(n\hat{x})]\}\,, \\
\mathrm{Tr}\,\tilde{V}_{2,+}(n\hat{y}+N\hat{z})\tilde{V}^{\dagger}_{2,+}(n\hat{y}+N\hat{x}) &\propto \exp\{i\,[\theta((n+1)\hat{y})-\theta(n\hat{y})]\}\,,
\end{aligned} \tag{5.14}$$

where $\hat{\Phi}_1 + i\hat{\Phi}_2 \propto \exp(i\theta)$. In eq. (5.8) all other phases cancel except those at the corners, and we obtain

$$\text{flux} = \arg\exp\{i\,[\theta(N\hat{y})-\theta(N\hat{x})]\} = \arg(-1) = \pi\,, \tag{5.15}$$

where we have used the boundary conditions for $\hat{\Phi}$ in eq. (5.6).

Thus we have shown that with these boundary conditions, there will be a flux π through each of the halves of the boundary, and since the boundary conditions force the direction of the flux to be opposite, this means a total flux of 2π from the lattice. Note that because the flux is only defined modulo 2π, it is only possible to force the total magnetic charge to be either even or odd.

5.2 Monopole mass

Using the boundary conditions (5.6), we can now define the effective mass of an isolated monopole. For any boundary conditions, the free energy is defined by

$$F = -\ln \int DU_i D\Phi \exp\left(-\sum_{\vec{x}} \mathcal{L}_{1\mathrm{H}}(\vec{x})\right). \tag{5.16}$$

If we denote by F_0 and F_1 the free energies for ensembles with boundary conditions (5.3) and (5.6), respectively, we can use eq. (5.2) directly to define the monopole mass M.

In order to write M as an expectation value, we will now transform the one-monopole system in such a way that the difference is moved from the boundary conditions into a shift in the action. Let us consider a field configuration that satisfies the conditions (5.6), and apply the following (large) gauge transformation

$$\Lambda = \begin{cases} 1\,, & \text{if } x, z < N \\ i\sigma_1\,, & \text{if } x < N\,, \quad z \geq N \\ i\sigma_3\,, & \text{if } x \geq N\,, \quad z < N \\ i\sigma_2\,, & \text{if } x, z \geq N\,. \end{cases} \tag{5.17}$$

This will change the boundary conditions into

$$\Phi(\vec{x}+N\hat{\jmath}) = -\sigma_2\Phi(\vec{x})\sigma_2 = \Phi^*(\vec{x})\,, \qquad U_k(\vec{x}+N\hat{\jmath}) = \sigma_2 U_k(\vec{x})\sigma_2 = U_k^*(\vec{x})\,, \tag{5.18}$$

everywhere else except at the edges of the lattice, where

$$\begin{aligned} U_3(x,N,N-1) &= -U_3{}^*(x,0,N-1)\,, \\ U_1(N-1,N,z) &= -U_1{}^*(N-1,0,z)\,, \\ U_1(N-1,y,N) &= -U_1{}^*(N-1,y,0)\,. \end{aligned} \tag{5.19}$$

The C-periodic boundary conditions (5.18) have been discussed before in ref. [32] in a different context.

When calculating the free energy (5.16), we can change the boundary conditions everywhere to eq. (5.18) by redefining

$$\begin{aligned} U_3(x,N,N-1) &\longrightarrow -U_3(x,N,N-1)\,, \\ U_1(N-1,N,z) &\longrightarrow -U_1(N-1,N,z)\,, \\ U_1(N-1,y,N) &\longrightarrow -U_1(N-1,y,N)\,. \end{aligned} \tag{5.20}$$

This changes the Wilson term in in the lagrangian (5.1). The total effect is to flip the sign of the Wilson term at three of the edges, and consequently the one-monopole free energy is given by

$$F_1 = -\ln \int DU_i D\Phi \exp\left(-\sum_{\vec{x}} \mathcal{L}_{1\mathrm{H}}(\vec{x}) - \Delta S\right), \tag{5.21}$$

with C-periodic boundary conditions (5.18). The change of the action is

$$\Delta S = \frac{4}{ag^2}\,\mathrm{Re}\left(\sum_{x=0}^{N-1}\mathrm{Tr}\,U_{23}(x,N-1,N-1)+\sum_{y=0}^{N-1}\mathrm{Tr}\,U_{13}(N-1,y,N-1)+\right.$$
$$\left.+\sum_{z=0}^{N-1}\mathrm{Tr}\,U_{12}(N-1,N-1,z)\right). \tag{5.22}$$

Because these boundary conditions preserve translation invariance, we could as well choose any $\vec{x}_0=(x_0,y_0,z_0)$ and write

$$\Delta S = \frac{4}{ag^2}\,\mathrm{Re}\left(\sum_{x=0}^{N-1}\mathrm{Tr}\,U_{23}(x,y_0,z_0)+\sum_{y=0}^{N-1}\mathrm{Tr}\,U_{13}(x_0,y,z_0)+\right.$$
$$\left.+\sum_{z=0}^{N-1}\mathrm{Tr}\,U_{12}(x_0,y_0,z)\right). \tag{5.23}$$

Note how similar this expression is to eq. (4.15). The only difference is that we have three intersecting 't Hooft loops instead of just one. Again, we emphasize that, because eq. (5.21) is equivalent to eq. (5.16) with the translation invariant boundary conditions eq. (5.6), the choice of $\vec{x}_0$ does not affect any observable, and in particular, it does not fix the location of the monopole on the lattice.

Without ΔS, eq. (5.21) would be precisely the ordinary free energy (5.16) with C-periodic boundary conditions (5.18). As they obviously imply

$$\tilde{V}_{i,+}(\vec{x}+N\hat{\jmath}) = \sigma_2\tilde{V}_{i,-}(\vec{x})\sigma_2\,, \tag{5.24}$$

the flux through the curve in figure 2 vanishes (cf. eq. (5.11)). Consequently, we can instead of the periodic boundary conditions (5.6), use the boundary conditions in eq. (5.18) to define F_0. This leads to the expression

$$M = -\ln\langle\exp(-\Delta S)\rangle\,, \tag{5.25}$$

where ΔS is given by eq. (5.23) and the boundary conditions by eq. (5.18).

To measure M, we can use the derivative with respect to m^2,

$$\frac{\partial M}{\partial m^2} = V\left(\langle\mathrm{Tr}\,\Phi^2\rangle_1-\langle\mathrm{Tr}\,\Phi^2\rangle_0\right), \tag{5.26}$$

where the subscript tells whether the expectation value is calculated in the zero-monopole or one-monopole system, and $V=a^3N^3$ is the volume of the system. If we assume that at large enough m^2, i.e. in the symmetric phase, the monopole mass vanishes, we only have to measure $\partial M/\partial m^2$ at sufficiently many values of m^2 and integrate.

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Like in the case of a vortex, we can also define for $\epsilon \in [0,1]$

$$F_\epsilon = -\ln \int DU_i D\Phi \exp\left(-\sum_{\vec{x}} \mathcal{L}_{1\mathrm{H}}(\vec{x}) - \epsilon \Delta S \right) . \tag{5.27}$$

Using eq. (5.27), we can now write the monopole mass as

$$M = \int_0^1 d\epsilon \frac{\partial F_\epsilon}{\partial \epsilon} = \int_0^1 d\epsilon \langle \Delta S \rangle_\epsilon \,, \tag{5.28}$$

where the subscript ϵ indicates that the expectation value must be measured in the presence of the insertion $-\epsilon\Delta S$ as in eq. (5.27). This gives us the absolute value of M, but with the cost that we have to measure expectation values at non-physical values of ϵ.

6. 4D pure gauge SU(2)

Because of the large amount of literature on condensation of monopoles as an explanation of confinement in pure-gauge SU(N) theory [9, 10, 18, 22, 23], it is interesting to see whether our formulation can be applied to this case as well.

The lagrangian of the 4D pure gauge theory is

$$\mathcal{L}_{4\mathrm{D}} = \beta \sum_{\mu<\nu} \left(1 - \frac{1}{2} \mathrm{Re}\ \mathrm{Tr}\, U_{\mu\nu}(t,\vec{x}) \right) . \tag{6.1}$$

In order to define monopoles, we will also have to specify the abelian projection [19] we are using, i.e. how Φ is related to the gauge fields. The original suggestion by 't Hooft [19] was to choose an operator $\mathcal{O}$ that is a product of the link variables and to define $\mathcal{O} = \mathcal{O}_0 + i\Phi$, where $\mathcal{O}_0$ is proportional to the unit matrix.

In order to use our definition for the monopole mass, the boundary conditions must be chosen in such a way that the induced boundary condition for Φ is compatible with eq. (5.6). However, for most choices of $\mathcal{O}$ this is not possible at all. This is obviously the case with all definitions of $\mathcal{O}$ that are products of spatial links U_i, because then the boundary conditions (5.6) imply

$$\mathcal{O}(t,\vec{x}+N\hat{\imath}) = \sigma_i \mathcal{O}(t,\vec{x})\sigma_i \,, \qquad \text{i.e.} \qquad \Phi(t,\vec{x}+N\hat{\imath}) = \sigma_i \Phi(t,\vec{x})\sigma_i \,, \tag{6.2}$$

in contradiction with eq. (5.6). As a consequence, these abelian projections are also incompatible with the classical monopole solution (5.4).

There are suitable ways of defining Φ, though, and one of them is the most popular choice of an abelian projection, namely the maximally abelian gauge [19, 23]. It means finding the gauge transformation $\Lambda(t,\vec{x})$ that maximizes the functional

$$R_{\mathrm{MAG}} = \sum_{\mu,t,\vec{x}} \mathrm{Tr}\, \sigma^3 \tilde{U}_\mu(t,\vec{x}) \sigma^3 \tilde{U}_\mu^\dagger(t,\vec{x}) \,, \tag{6.3}$$

where $\tilde{U}_i(t,\vec{x}) = \Lambda^\dagger(t,\vec{x})U_i(t,\vec{x})\Lambda(t,\vec{x}+\hat{\imath})$ and $\tilde{U}_0(t,\vec{x}) = \Lambda^\dagger(t,\vec{x})U_0(t,\vec{x})\Lambda(t+1,\vec{x})$. Defining $\hat{\Phi}(t,\vec{x}) = \Lambda(t,\vec{x})\sigma^3\Lambda^\dagger(t,\vec{x})$, we can write eq. (6.3) as

$$R_{\rm MAG} = \sum_{t,\vec{x}}\Bigg[\mathrm{Tr}\,\hat{\Phi}(t,\vec{x})U_0(t,\vec{x})\hat{\Phi}(t+1,\vec{x})U_0^\dagger(t,\vec{x}) + \\ + \sum_i \mathrm{Tr}\,\hat{\Phi}(t,\vec{x})U_i(t,\vec{x})\hat{\Phi}(t,\vec{x}+\hat{\imath})U_i^\dagger(t,\vec{x})\Bigg], \tag{6.4}$$

which must be maximized with respect to the field $\hat{\Phi}(t,\vec{x})$. It is obvious that if $\hat{\Phi}$ maximizes $R_{\rm MAG}$ in the region $\vec{x}\in\{0,\ldots,N-1\}^3$ and U_μ and $\hat{\Phi}$ satisfy the generalization of the boundary conditions (5.6),

$$\hat{\Phi}(t,\vec{x}+N\hat{\jmath}) = -\sigma_j\hat{\Phi}(t,\vec{x})\sigma_j\,, \qquad U_\mu(t,\vec{x}+N\hat{\jmath}) = \sigma_j U_\mu(t,\vec{x})\sigma_j\,, \tag{6.5}$$

then $\hat{\Phi}$ maximizes $R_{\rm MAG}$ everywhere. Thus the maximally abelian gauge is compatible with the boundary conditions (5.6).

We will now assume that Φ is given either by the maximally abelian gauge condition or some other way that is compatible with eq. (6.5). The definition of the magnetic field can be generalized from eq. (3.15) trivially by simply using the same definition at every time slice. Consequently, we can perform the gauge transformation (5.17) and the field redefinition (5.20) or equivalently simply take eq. (5.25) and write

$$M = -\lim_{L_t\to\infty}\frac{1}{L_t}\exp\left(-\sum_t \Delta S(t)\right), \tag{6.6}$$

where the boundary conditions in the spatial directions are C-periodic (5.18), L_t is the length of the lattice in the time direction and

$$\Delta S(t) = \beta\,\mathrm{Re}\left(\sum_{x=0}^{N-1}\mathrm{Tr}\,U_{23}(t,x,y_0,z_0) + \sum_{y=0}^{N-1}\mathrm{Tr}\,U_{13}(t,x_0,y,z_0) + \right. \\ \left. + \sum_{z=0}^{N-1}\mathrm{Tr}\,U_{12}(t,x_0,y_0,z)\right). \tag{6.7}$$

As before, the mass can be measured in practice using the multi-histogram techniques [14, 27, 28] or plaquette by plaquette [16]. Note that eq. (6.6) does not refer to Φ at all and therefore, the tedious task of numerically maximizing $R_{\rm MAG}$ is avoided altogether. Moreover, as long as the definition of Φ is compatible with eq. (6.5), the mass of a monopole is independent of the precise choice of the abelian projection.

Let us now compare our prescription with that used in ref. [9]. There, instead of the monopole mass, the disorder parameter $\langle\mu\rangle$ was measured from the correlator $\langle\bar{\mu}(t_i)\mu(t_f)\rangle$. The correlator was defined by shifting the vertical plaquettes U_{i0} in the action at $t=t_i$

$$\begin{aligned} U_{i0}(t_i,\vec{x}) &= U_i(t_i,\vec{x})U_0(t_i,\vec{x}+\hat{\imath})U_i^\dagger(t_i+1,\vec{x})U_0^\dagger(t_i,\vec{x}) \\ &\longrightarrow U_i(t_i,\vec{x})U_0(t_i,\vec{x}+\hat{\imath})U_i^\dagger(t_i+1,\vec{x})U_{\mathrm{cl},i}^\dagger(t_i+1,\vec{x})U_0^\dagger(t_i,\vec{x})\,, \end{aligned} \tag{6.8}$$

where $U_{\mathrm{cl},i}$ is the gauge field of a static classical abelian monopole, and correspondingly at $t=t_f$. By redefining the spatial links U_i,

$$U_{\mathrm{cl},i}(t,\vec{x})U_i(t,\vec{x}) \longrightarrow U_i(t,\vec{x})\,, \tag{6.9}$$

the shift can be moved to the spatial plaquettes, thus making direct comparison possible. Leaving aside the problems caused by the fact that the redefinition (6.9) changes the boundary conditions and with most choices of the abelian projection also the value of Φ, we can write the resulting shift in the action as

$$\Delta S(t) = -\beta\sum_{\vec{x}}\sum_{i<j}\mathrm{Re}\left(\frac{1}{2}\,\mathrm{Tr}\,U'_{ij}(t,\vec{x}) - \frac{1}{2}\,\mathrm{Tr}\,U_{ij}(t,\vec{x})\right), \tag{6.10}$$

where $U'_i = U_{\mathrm{cl},i}^\dagger U_i$.

The structure of eqs. (6.7) and (6.10) is similar, and realizing that in the full non-perturbative treatment there is no reason for $U_{\mathrm{cl},i}$ to be a solution of the classical field equations, we can interpret eq. (6.7) as simply a particularly convenient choice of $U_{\mathrm{cl},i}$. For instance, it makes it much easier to choose suitable boundary conditions that preserve the translation invariance and allow a non-zero total magnetic charge.

The measurement of the monopole mass has also been discussed in ref. [22]. There the non-zero total magnetic charge was achieved by using fixed boundary conditions. However, this breaks the translation invariance, and boundary effects can therefore be significant.

7. Conclusions

In this paper, we have presented a way of measuring free energies of topological defects in SU(2) gauge theories on a lattice. Our approach was to impose "twisted" boundary conditions, which preserve the translation invariance and minimize boundary effects but guarantee that the number of topological defects inside the system is non zero. By changing the integration variables in the partition function, the free energy of the defect can then be expressed as an expectation value, which makes it possible to measure it in numerical simulations. In the case of the two-Higgs model, we derived an expression for the vortex tension in terms of a 't Hooft loop using this technique.

The main result of this paper is the derivation of an analogous expression for the mass of a 't Hooft-Polyakov monopole in the Georgi-Glashow model, defined

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as the free energy difference between configurations of total magnetic charge one and zero. Again, the construction was based on manifestly translation invariant boundary conditions, which guarantees that the boundary effects are minimal. By a gauge transformation and a redefinition of the fields, we rewrote the free energy as a path integral with C-periodic boundary conditions and with an insertion ΔS, which consists of three intersecting 't Hooft loops. In addition to avoiding problems with boundary conditions, our prescription leads to a simpler final expression for the monopole mass than the ones appeared previously in the literature.

We believe that the monopole mass (5.25) can be used to distinguish between the phases of the Georgi-Glashow model, because it is zero in the SU(2) phase and non zero in the U(1) phase. More generally, we conjecture that if the symmetry breaking structure of a phase transition in a gauge theory allows topological defects and it is possible to construct an observable that measures the free energy of such a defect in the same way as eqs. (4.16) and (5.25) measure the free energies of vortices and monopoles, the point at which the defect free energy becomes non-zero defines a transition point between the phases. Although this does not necessarily imply a singularity in the partition function, it suggests that in these cases the phase transition is not likely to be a smooth crossover.

We also showed that the same construction can be used in pure-gauge theory to observe condensation of abelian monopoles. In this framework, it is not necessary to fix the gauge or carry out the abelian projection in the simulation. This simplifies the procedure significantly and means that the observable, and therefore also the resulting monopole mass, is manifestly independent of the choice of the abelian projection, provided that it is compatible with the classical 't Hooft-Polyakov monopole solution. Thus, the ambiguities related to the differences of various abelian projections can be avoided by adopting eq. (6.6) as the definition of condensation of abelian monopoles.

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Acknowledgments

The authors wish to thank K. Rummukainen for generously providing the code which acted as a template for the two Higgs calculations. This work was supported in part by PPARC and an ESF network. H.P.S. was supported by the JSPS and A.R. partially by the University of Helsinki. The computational work for this paper was carried out on a Silicon Graphics Origin 2000 supercomputer of the UK Computational Cosmology Consortium based at the University of Cambridge.

A. Computational details of the two Higgs model calculation

The configurations for this calculation were generated using a combination of the Kennedy-Pendleton [33] heatbath algorithm and the over-relaxed [34, 35] algorithm

V/a^3	Number of thermalization steps	Number of measurements
12^3	50,000	200
16^3	50,000	200
24^3	10,000	40

Table 1: Run parameters for lattice sizes presented in figure 1.

along with a Metropolis accept-reject step for the pure gauge sector and the scalar-gauge interactions and a Metropolis step for the scalar interactions. Measurements are carried out every 500 compound sweeps. In table 1, we list the thermalization and number of measurements carried out for the lattices used in figure 1.

As outlined in [36], if we define

$$\rho_{(i)}(\vec{x}) = a\,\mathrm{Tr}\,\chi^2_{(i)}\,, \tag{A.1}$$

where i indicates the i-th configuration, (the fields computed on the lattice are $\sqrt{a}\chi$) and

$$\overline{\rho_i} = \frac{V}{a^3}\sum_{\vec{x}} \rho_{(i)}(\vec{x})\,, \tag{A.2}$$

then the susceptibility in figure 1 is defined as

$$S(\chi) = \frac{V}{a^3}\left(\frac{\langle\overline{\rho_i}\overline{\rho_i}\rangle}{\langle\overline{\rho_i}\rangle^2} - 1\right), \tag{A.3}$$

where the notation $\langle\mathcal{O}\rangle$ represents an average over configurations.

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